THREE LECTURES ON QUASIDETERMINANTS

Robert Lee Wilson

Department of Mathematics

Rutgers University

The **determinant** of a matrix with entries in a commutative ring is a main organizing tool in commutative algebra. In these lectures, we present an analogous theory, the theory of **quasideterminants**, for matrices with entries in a not necessarily commutative ring. The theory of quasideterminants was originated by I. Gelfand and V. Retakh.

Outline:

Lecture 1:

a) What are quasideterminants?

b) Comparison with the commutative determinants

c) Effect of row and column operations on quasideterminants

d) Applications to linear systems

Lecture 2:

a) Quasideterminants are, in general, quite complicated: Reutenauer's Theorem on inversion height

b) Some quasideterminants can be expressed as polynomials

c) Quasideterminants of block matrices

d) Noncommutative determinants (in terms of quasideterminants)

e) Previously known theories of noncommutative determinants

f) Determinants of quantum matrices

g) Determinants of quaterionic matrices

h) Dieudonne determinants

Lecture 3:

a) Application of quasideterminants to the study of polynomial equations - the noncommutative Viete Theorem of Gelfand and Retakh

b) Elementary symmetric functions - the fundamental theorem

c) Complete symmetric functions

d) Power sum symmetric functions

References:

1) I. Gelfand and V. Retakh, Determinants of Matrices over Noncommutative Rings, Funct. Anal. Appl **25** (1991), no. 2, 91-102.

2) I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, Quasideterminants, J. Algebra **193**, 2005, 56-141.

3) B. Osofsky, Quasideterminants and an interesting algebra they spawned, http://www.math.rutgers.edu/ osofsky/AthensSlides05.pdf

Reference (1) is the first paper on the subject. Reference (2) is a comprehensive survey article with an extensive bibliography. Reference (3) presents a particularly useful way of stating the definition of a quasideterminant.

Lecture 1:

a) What are quasideterminants?

An n by n matrix A over a not necessarily commutative ring R has, in general, n^2 quasideterminants, denoted $[A]_{i,j}, 1 \leq i, j \leq n$. If $[A]_{i,j}$ exists it is a rational function in the entries of A. Here are several essentially equivalent definitions of $[A]_{i,j}$.

(1) Definition of quasideterminant via Gaussian elimination (following B. Osofsky):

Suppose the matrix A can be transformed into a matrix of the form

$$\begin{bmatrix} I_{n-1} & b \\ 0 & c \end{bmatrix}$$

by a sequence of elementary row operations which do not interchange rows, which do not multiply the n-th row by a scalar and which do not add add multiples of the n-th row to any other row, i.e., by operations of the following forms:

• multiply each entry of the i-th row (where $1 \leq i < n$) on the left by some $r \in R$ and leave all other rows unchanged;

• replace the i-th row (where $1 \leq i \leq n$) by the sum of the i-th row and the j-th row (where $1 \leq j < n$) and leave all other rows unchanged.

(These operations allow us to replace A by MA where $M = [m_{i,j}]$ is an n by n matrix with $m_{i,n} = \delta_{i,n}$ for $1 \leq i \leq n$.)

Then the quasideterminant $[A]_{n,n}$ exists and

$$[A]_{n,n} = c.$$

If P and Q are the permutation matrics corresponding to the transpositions (in) and (jn) respectively and $[PAQ]_{n,n}$ exists, then $[A]_{i,j}$ exists and

$$[A]_{i,j} = [PAQ]_{n,n}.$$

(2) Definition of quasideterminants of A via the inverse of A:

Suppose the matrix A is invertible with inverse $B = [b_{i,j}]$ and that $b_{j,i}$ is invertible in R. Then the quasideterminant $[A]_{i,j}$ exists and

$$[A]_{i,j} = b_{j,i}^{-1}.$$

(3) Definition of quasideterminants of A via inverses of minors of A:

For $1 \leq i, j \leq n$ let $A^{i,j}$ denote the matrix obtained from A by deleting the i-th row and the j-th column. Let r_k^j denote the (row) vector obtained from the k-th row of A by deleting the j-th entry and let s_l^i denote the (column) vector obtained from the l-th column of A by deleting the i-th entry. Assume that $A^{i,j}$ is invertible. Then $[A]_{i,j}$ exists and

$$[A]_{i,j} = a_{i,j} - r_i^j (A^{i,j})^{-1} s_j^i.$$

(4) Inductive definition of quasideterminant:

If $A = [a]$ is a 1 by 1 matrix define $[A]_{1,1} = a$. Assume quasideterminants have been defined for $n-1$ by $n-1$ matrices and that $A = [a_{i,j}]$ is an n by n

matrix. Using the notation of (3), assume that each quasideterminant of $A^{i,j}$ exists and is invertible. Let C denote the $n-1$ by $n-1$ matrix with rows indexed by $\{k|1 \leq k \leq n, k \neq i\}$ and columns indexed by $\{l|1 \leq l \leq n, l \neq j\}$ and whose entry in the (k,l) position is $[A^{i,j}]^{-1}_{l,k}$. Then the quasideterminant $[A]_{i,j}$ exists and

$$[A_{i,j}] = a_{i,j} - r_i^j C s_j^i.$$

Equivalence of these definitions:

Suppose $MA = \begin{bmatrix} I_{n-1} & b \\ 0 & c \end{bmatrix}$ where M is a product of appropriate elementary matrices. Then if c is invertible

$$\begin{bmatrix} I_{n-1} & -bc^{-1} \\ 0 & 1 \end{bmatrix} MA = \begin{bmatrix} I_{n-1} & 0 \\ 0 & c \end{bmatrix}$$

and so, c^{-1} is the entry (n,n) entry of A^{-1}. This shows the equivalence of (1) and (2).

If we write the n by n matrix A as the block matrix $\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$ where $A_{1,1}$ is an $n-1$ by $n-1$ matrix and assume that $A_{1,1}$ is invertible, then

$$\begin{bmatrix} I_{n-1} & 0 \\ -A_{2,1} & I_1 \end{bmatrix} \begin{bmatrix} A_{1,1}^{-1} & 0 \\ 0 & I_1 \end{bmatrix} A = \begin{bmatrix} I_{n-1} & A_{1,1}^{-1}A_{1,2} \\ 0 & A_{22} - A_{2,1}A_{1,1}^{-1}A_{1,2} \end{bmatrix}.$$

Since $A_{1,1} = A^{n,n}, A_{2,1} = r_n^n, A_{1,2} = s_n^n$, this shows the equivalence of (1) and (3). The equivalence of (3) and (4) now follows from (2).

If A is the 2 by 2 matrix $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ then we have

$$[A]_{1,1} = a_{1,1} - a_{1,2}a_{2,2}^{-1}a_{2,1},$$

$$[A]_{1,2} = a_{1,2} - a_{1,1}a_{2,1}^{-1}a_{2,2},$$

$$[A]_{2,1} = a_{2,1} - a_{2,2}a_{1,2}^{-1}a_{1,1},$$

$$[A]_{2,2} = a_{2,2} - a_{2,1}a_{1,1}^{-1}a_{1,2},$$

b) Comparison with the commutative determinants

Suppose A is a matrix over a commutative ring R. How is the quasideterminant $[A]_{i,j}$ related to $det\ A$?

It is well known that, if A is invertible, the (j, i) entry of A^{-1} is $(-1)^{i+j}\frac{det\ A^{i,j}}{det\ A}$.

Thus, in view of characterization (2) of $[A]_{i,j}$ we have

$$[A]_{i,j} = (-1)^{i+j}\frac{det\ A}{det\ A^{i,j}}.$$

c) Effect of row and column operations on quasideterminants

We now describe the effect of certain row and column operations on the values of quasideterminants. This is useful for comparison with properties of determinants over commutative rings and will also be used for some computations in later lectures.

(i) The quasideterminant $[A]_{pq}$ does not depend on permutations of rows and columns in the matrix A that do not involve the p-th row and the q-th column.

(ii) *The multiplication of rows and columns.* Let the matrix $B = [b_{ij}]$ be obtained from the matrix A by multiplying the i-th row by $\lambda \in R$ from the left, i.e., $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for
$k \neq i$. Then

$$[B]_{kj} = \lambda [A]_{ij} \text{ if } k = i,$$

and

$$[B]_{kj} = [A]_{kj} \text{ if } k \neq i \text{ and } \lambda \text{ is invertible.}$$

Let the matrix $C = [c_{ij}]$ be obtained from the matrix A by multiplying the j-th column by $\mu \in R$ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all i and $l \neq j$. Then

$$[C]_{i\ell} = [A]_{ij}\mu \text{ if } l = j,$$

and

$$[C]_{i\ell} = [A]_{i\ell} \text{ if } l \neq j \text{ and } \mu \text{ is invertible.}$$

(iii) *The addition of rows and columns.* Let the matrix B be obtained from A by replacing the k-th row of A with the sum of the k-th and l-th rows, i.e., $b_{kj} = a_{kj} + a_{lj}$, $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$[A]_{ij} = [B]_{ij}, \qquad i = 1, \ldots k-1, k+1, \ldots n, \quad j = 1, \ldots, n.$$

Let the matrix C be obtained from A by replacing the k-th column of A with the sum of the k-th and l-th columns, i.e., $c_{ik} = a_{ik} + a_{il}$, $c_{ij} = a_{ij}$ for $j \neq k$. Then

$$[A]_{ij} = [C]_{ij}, \qquad i = 1, \ldots, n, \quad , \ldots, \ell - 1, \ell + 1, \ldots n.$$

d) Applications to linear systems

Solutions of systems of linear systems over an arbitraty ring can be expressed in terms of quasideterminants.

Let $A = [a_{ij}]$ be an $n \times n$ matrix over a ring R.

Theorem: Assume that all the quasideterminants $[A]_{ij}$ are defined and invertible. Then the system of equations

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i,\ 1 \leq i \leq n$$

has the unique solution

$$x_i = \sum_{j=1}^{n} [A]_{ji}^{-1} b_j. \qquad i = 1, \ldots, n.$$

Proof: In view of characterization (2) of quasideterminants, the assumption that every $[A]_{i,j}$ is defined and invertible implies that A is invertible. The result now follows by using A^{-1} to write the solution of the system and replacing the elements of A^{-1} by quasideterminants.

This is a short selection of properties of quasideterminants. Many other properties are described in reference (2).

Lecture 2:

a) In general, quasideterminants are quite complicated. For example, from the definition

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}_{1,1} =$$

$$= a_{1,1} - a_{1,2} \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}_{2,2}^{-1} a_{2,1}$$

$$-a_{1,3} \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}_{2,3}^{-1} a_{2,1}$$

$$-a_{1,2} \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}_{3,2}^{-1} a_{3,1}$$

$$-a_{1,3} \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}_{3,3}^{-1} a_{3,1}$$

$$= a_{1,1} - a_{1,2}(a_{2,2} - a_{2,3}a_{3,3}^{-1}a_{3,2})^{-1}a_{2,1}$$

$$-a_{1,3}(a_{2,3} - a_{2,2}a_{3,2}^{-1}a_{3,3})^{-1}a_{2,1}$$

$$-a_{1,2}(a_{3,2} - a_{3,3}a_{2,3}^{-1}a_{2,2})^{-1}a_{3,1}$$

$$-a_{1,3}(a_{3,3} - a_{3,2}a_{2,2}^{-1}a_{2,3})^{-1}a_{3,1}.$$

Notice that the last four terms involve two "nested" inversions that is, there is an inverse inside the parentheses and the quantity in parentheses is inverted.

¿From the inductive definition of $[A]_{i,j}$ in appears that if$A$ is an n by n matrix, then $[A]_{i,j}$ will involve $n-1$ nested inversions.

Of course, it might be possible to write the quasideterminant in some other way to avoid having this many nested inversions. To discuss whether this is possible or not, we need some additional definitions.

Let R be a subring of a division ring D. Assume that R generates D as a division ring; that is, assume that there is no proper sub-division ring of D that contains R.

Set $R_0 = R$ and, for $n \geq 0$, let R_{n+1} denote the subring of D generated by

$$R_n \cup \{r^{-1} | 0 \neq r \in R_n\}.$$

Then $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq ...$ and if $R_i = R_{i+1}$ then R_i is a division ring, so $R_i = D$.

If $R_n = D, R_{n-1} \neq D$ we say D has height n over R. (For example, if R is a (commutative) integral domain but not a field and D is its quotient field, then D has height 1 over R.) J. L. Fisher (Proc. Amer. Math. Soc. **30** (1971), 453-458) has shown that if $R = F < x_1, x_2 >$, the free algebra on x_1, x_2, then there are division rings of height 1 and 2 over R.

If S is the free division ring on $x_1, ..., x_n$ where $n \geq 2$, we say that $u \in S$ has height n if $u \in R_n, u \notin R_{n-1}$.

The following theorem is my candidate for the deepest result anyone has yet proved about quasideterminants.

Theorem (Reutenauer): If $A = [x_{i,j}]$ is an n by n matrix over the free division ring on $\{x_{i,j} | 1 \leq i, j \leq n\}$, then $[A]_{k,l}$ has height $n-1$ over

$F < x_{i,j} | 1 \leq i, j \leq n >$.

b) Sometimes quasideterminants are just polynomials.

Proposition:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & ... & a_{1,n-1} & a_{1,n} \\ -1 & a_{2,2} & a_{2,3} & ... & a_{2,n-1} & a_{2,n} \\ 0 & -1 & a_{3,3} & ... & a_{3,n-1} & a_{3,n} \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & -1 & a_{n,n} \end{bmatrix}_{1,n}$$

is a polynomial in the $a_{i,j}$. In fact it is

$$a_{1,n} + \sum a_{1,j_1} a_{j_1+1,j_2} a_{j_2+1,j_3} ... a_{j_{k+1},n}$$

where the sum is over all $1 \leq j_1 < j_2 < ... < j_k < n$.

To see that this quasideterminant is a polynomial, write the quasideterminant as

$$a_{1,n} - uCv$$

where

$$u = [a_{1,1}, a_{1,2}, ..., a_{1,n-1}]$$

$$C = \begin{bmatrix} -1 & a_{2,2} & a_{2,3} & ... & a_{2,n} \\ 0 & -1 & a_{3,3} & ... & a_{3,n} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & -1 \end{bmatrix}^{-1}$$

and

$$v = \begin{bmatrix} a_{2,n} \\ a_{3,n} \\ . \\ . \\ . \\ a_{n,n} \end{bmatrix}.$$

Now

$$C = -(I - N)^{-1}$$

where

$$N = \begin{bmatrix} 0 & a_{2,2} & a_{2,3} & ... & a_{2,n} \\ 0 & 0 & a_{3,3} & ... & a_{3,n} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & a_{n-1,n-1} \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}.$$

Since N is nilpotent (in fact, $N^n = 0$) we have

$$(I - N)^{-1} = I + N + N^2 + ... + N^{n-1}.$$

Thus the entries of $(I-N)^{-1}$ are polynomials in the $a_{i,j}$ and the fact that the quasideterminant is a polynomial in its entries follows (as does the formula for the value if we write out $(I-N)^{-1}$).

c) Quasideterminants of block matrices

Theorem: Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

be a block decomposition of an n by n matrix A where $A_{1,1}$ is k by k and $A_{2,2}$ is invertible. If $1 \leq i, j \leq k$ and if either one of

$$[A]_{i,j}$$

and

$$[A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1}]_{i,j}$$

is defined, then both are defined and they are equal.

(For a more general version of this result, see Theorem 1.4.3 of reference (2).)

d) Noncommutative determinants (in terms of quasideterminants)

We have seen that for matrices over a commutative ring, the quasideterminant $[A]_{i,j}$ is not the determinant, but rather a ratio of determinants:

$$[A]_{i,j} = (-1)^{i+j}\frac{det\ A}{det\ A^{i,j}}.$$

In this case we may recover the determinant from quasideterminants. Thus

$$det\ A = [A]_{1,1}[A^{1,1}]_{2,2}[A^{12,12}]_{3,3}...[A^{12...(n-1),12...(n-1)}]_{n,n}.$$

(Of course we could replace this sequence of minors $A^{1,1}, A^{12,12}, ...$ by any other nested sequence of minors of sizes $n-1, n-2, ...$ and obtain $det\ A$ or $-det\ A$.)

With this as motivation, we define, if $I = \{i_1, ..., i_n\}$ and $J = \{j_1, ..., j_n\}$ are two orderings of $\{1, ..., n\}$ and A is an n by n matrix over a not necessarily commutative ring R

$$D_{I,J}(A) = [A]_{i_1,j_1}[A^{i_1,j_1}]_{i_2,j_2}[A^{i_1 i_2, j_1 j_2}]_{i_3,j_3}...[A^{i_1 i_2 ... i_{(n-1)}, j_1 j_2 ... j_{(n-1)}}]_{i_n,j_n}.$$

Note that the last factor is a_{i_n,j_n}. We compare $D_{I,J}$ with some previously known definitions of noncommutative determinants.

e) Previously known theories of noncommutative determinants

Attempts to define determinants over noncommuttive rings date, at least, to a paper of Cayley's in 1845. A partial list of constructions of noncommutative determinants includes work of Berezin,Birman-Williams, Capelli, Cartier-Foata, Cayley, Draxl, Kirillov, Moore, Dieudonne, Dyson, Heyting, Richardson, Study, and Wedderburn.

In general, these authors use one of the equivalent characterizations of determinants in the commutative case as the definition. Thus there are definitions generalizing

- $\det A = \sum sgn(\sigma) a_{1\sigma(1)} ... a_{n\sigma(n)}$,
- $\det A = \text{product of eigenvalues}$,
- $A = LDU, \det A = \text{product of diagonal entries of } D$,
- transformation properties of $\det A$ under elementary row operations.

In order to prove useful results, the authors restrict consideration to certain rings (usually "close" to commutative) and perhaps also to certain classes of matrices (e.g., Hermitian quaterionic matrices). Three specific examples follow:

f) Determinants of quantum matrices

Recall that $A = [a_{ij}], i, j = 1, \ldots, n$, is a *quantum* matrix over a field F if $q \in F$ is given such that

$$a_{ik}a_{il} = q^{-1}a_{il}a_{ik},\ if\ k < l,$$

$$a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik},\ if\ i < j,$$

$$a_{il}a_{jk} = a_{jk}a_{il},\ if\ i < j, k < l,$$

$$a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)a_{il}a_{jk},$$

$$if\ i < j, k < l.$$

The quantum determinant

$$\det_q A$$

is defined to be

$$\sum_{\sigma \in S_n} (-q)^{-l(\sigma)} a_{1\sigma(1)}a_{2\sigma(2)} \ldots a_{n\sigma(n)},$$

where $l(\sigma)$ is the number of inversions in σ.

If A is a quantum matrix then any square submatrix of A also is a quantum matrix with the same scalar q.

Theorem:

$$\det_q\ A = (-q)^{i-j}[A]_{ij} \cdot \det_q\ A^{ij} =$$

$$(-q)^{i-j}\det_q A^{ij}\cdot[A]_{ij}.$$

Corollary:

$$\det_q A=[A]_{11}[A^{11}]_{22}\dots a_{nn}$$

and all factors in the right-hand side commute.

g) Determinants of quaterionic matrices

A quaternionic matrix

$$A=[a_{ij}],$$

$i,j=1,\dots,n$, is called Hermitian if $a_{ij}=\bar{a}_{ji}$ for all i,j.

The notion of determinant for Hermitian quaternionic matrices

was introduced by E.M. Moore in 1922. Here is the original definition.

Let $A=(a_{ij})$, $i,j=1,\dots,n$, be a matrix over a ring. Let σ be a permutation of $\{1,\dots,n\}$. Write σ as a product of disjoint cycles. Since disjoint cycles commute, we may write

$$\sigma=(k_{11}\dots k_{1j_1})(k_{21}\dots k_{2j_2})$$

$$\dots(k_{m1}\dots k_{mj_m})$$

where for each i, we have $k_{i1}<k_{ij}$ for all $j>1$, and $k_{11}>k_{21}>\dots>k_{m1}$. This expression is unique. The Moore determinant $M(A)$ is defined as follows:

$$M(A)=$$

$$\sum_{\sigma\in S_n} sgn(\sigma)a_{k_{11},k_{12}}\dots a_{k_{1j_1},k_{11}}\cdot$$

$$a_{k_{21},k_{22}}\dots a_{k_{mj_m},k_{m1}}.$$

Theorem: Let A be a Hermitian quaternionic matrix. Then $M(A) = \pm[A]_{11}[A^{11}]_{22}...a_{nn}$.

h) Dieudonne determinants

Let R be a division algebra, $R^* = R \setminus \{0\}$ the monoid of invertible elements in R and

$$\pi : R^* \to R^*/[R^*, R^*]$$

the canonical homomorphism. To the abelian group $R^*/[R^*, R^*]$ we adjoin the zero element 0 with obvious multiplication, and denote the obtained semi-group by $\tilde{R}$. Extend π to a map $R \to \tilde{R}$ by setting $\pi(0) = 0$.

There exists a unique homomorphism

$$\det : M_n(R) \to \tilde{R}$$

such that

(i) $\det A' = \tilde{\mu} \det A$ for any matrix A' obtained from $A \in M_n(R)$ by multiplying one row of A from the left by μ;

(ii) $\det A'' = \det A$ for any matrix A'' obtained from A by adding one row to another;

(iii) $\det(E_n) = 1$ for the identity matrix E_n.

The homomorphism det is called the Dieudonne determinant.

It is known that $\det A = 0$ if $rank(A) < n$. If $rank(A) = n$ we have:

Proposition: Let A be a $n \times n$-matrix over a division algebra R. If $rank(A) = n$, then

(i) There exist orderings $I = \{i_1, ..., i_n\}$ and $J = \{j_1, ..., j_n\}$ of $\{1, \ldots, n\}$

such that

$$D_{I,J}(A) = [A]_{i_1 j_1}[A^{i_1 j_1}]_{i_2 j_2} ... a_{i_n j_n}$$

is defined.

(ii) If $D_{I,J}(A)$ is defined, then $\pi(D_{I,J}(A)) = \pm \det A$.

Lecture 3:

a) Application of quasideterminants to the study of polynomial equations - the noncommutative Viete Theorem of Gelfand and Retakh

One of the earliest theorems in algebra, due to Francois Viete (who also discovered the law of cosines), expresses the coefficients of a polynomial equation in terms of its roots: If the polynomial equation $f(x) = 0$, where $f(x)$ is monic of degree n over a (commutative) field, has n roots $x_1, ..., x_n$, then $f(x) = (x - x_1)...(x - x_n)$. Gelfand and Retakh have used the theory of quasideterminants to give a generalization to equations over noncommutative rings.

For quadratic equations this is not hard. If x_1 and x_2 are roots of $x^2 + a_1x + a_2 = 0$, then we have

$$x_1^2 + a_1x_1 + a_2 = 0$$

and

$$x_2^2 + a_1x_2 + a_2 = 0.$$

Taking the difference gives

$$x_1^2 - x_2^2 + a_1(x_1 - x_2) = 0.$$

This may be rewritten as

$$x_1(x_1 - x_2) + (x_1 - x_2)x_2 + a_1(x_1 - x_2) = 0$$

which gives

$$a_1 + x_1 = -(x_1 - x_2)x_2(x_1 - x_2)^{-1}$$

so that

$$a_1 = -x_1 - (x_1 - x_2)x_2(x_1 - x_2)^{-1}.$$

It is then easy to see that

$$a_2 = (x_1 - x_2)x_2(x_1 - x_2)^{-1}x_1.$$

In general, let $x_1, x_2, \ldots, x_n$ be a set of elements of a division algebra R. For $1 < k \leq n$ the quasideterminant

$$V(x_1, \ldots, x_k) = \begin{bmatrix} x_1^{k-1} & \ldots & x_k^{k-1} \\ & \ldots & \\ x_1 & \ldots & x_k \\ 1 & \ldots & 1 \end{bmatrix}_{1k}$$

is called the *Vandermonde* quasideterminant

We say that a sequence of elements $x_1, \ldots, x_n \in R$ is *independent* if all quasideterminants

$$V(x_1, \ldots, x_k),\ k = 2, \ldots, n,$$

are defined and invertible.

For an independent sequence

$$x_1, \ldots, x_n$$

set

$$y_1 = x_1$$

and

$$y_k = V(x_1, \ldots, x_k)x_kV(x_1, \ldots, x_k)^{-1}$$

for $2 \leq k \leq n$. Define

$$\Lambda_k(x_1, ..., x_n) =$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} y_{i_k} y_{i_{k-1}} \cdots y_1.$$

Theorem: (Noncommutative Vieta Theorem) Let $\{x_1, ..., x_n\}$ be an independent sequence of roots of the polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

over R. Then

$$a_i = (-1)^i \Lambda_i(x_1, ..., x_n)$$

for $1 \leq i \leq n$.

The y_i also provide information about factorization of polynomials in $R[t]$ (where t commutes with elements of R).

Theorem:

$$t^n + a_1 t^{n-1} + \cdots + a_n = (t - y_n)...(t - y_1).$$

b) Elementary symmetric functions - the fundamental theorem

Note that the a_i in the noncommutative Vieta Theorem are independent of the order of the x. Therefore, the $\Lambda_i(x_1, ..., x_n)$ are symmetric functions of $x_1, ..., x_n$. In fact:

Theorem: Any polynomial in $y_1, ..., y_n$ which is invariant under the action of the symmetric group (on $x_1, ..., x_n$) is a polynomial in

$$\{\Lambda_1(x_1, ..., x_n), ..., \Lambda_n(x_1, ..., x_n)\}.$$

This theorem is the analogue of the fundamental theorem on symmetric functions over a (commutative) field. The Λ_i are the analogues of the (commutative) elementary symmetric functions. It can be shown that the $\{y_1, ..., y_n\}$ is algebraically independent (in the sense that if $f(w_1, ..., w_n)$ is a polynomial in noncommutative variables $w_1, ..., w_n$ and $f(y_1, ..., y_n) = 0$ then $f(w_1, ..., w_n) = 0$). Thus the algebra generated by $y_1, ..., y_n$ over the base field F is free and consequently the algebra generated by $\Lambda_1, ..., \Lambda_n$ is free.

c) Complete symmetric functions

In the commutative case, other families of symmetric functions may be defined in terms of the elementary symmetric functions. Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon have used the same definitions, applied to the Λ_i to define families of noncommutative symmetric functions. Thus they set

$$\lambda(t) = 1 + \sum_{k>0} \Lambda_k t^k,$$

a formal power series in the commuting variable t over the free algebra generated by the Λ_k. They then define

$$\sigma(t) = \lambda(-t)^{-1}$$

and write

$$\sigma(t) = 1 + \sum_{k>0} \ S_k.$$

The S_k are the **complete symmetric functions**.

The coefficient of t^n in the generating function identity $\sigma(t) = \lambda(-t)^{-1}$ is 0 if $n > 1$ and so we have

$$S_n - S_{n-1}\Lambda_1 + S_{n-2}\Lambda_2 + ... + (-1)^n \Lambda_n = 0.$$

Solving the resulting linear system of n equations in $S_1, ..., S_n$ by using the quasideterminantal version of Cramer's rule gives

$$S_n = (-1)^{n-1} \begin{bmatrix} \Lambda_1 & \Lambda_2 & ... & \Lambda_n - 1 & \lambda_n \\ \Lambda_0 & \Lambda_1 & ... & \Lambda_{n-2} & \Lambda_{n-1} \\ 0 & \Lambda_0 & ... & \Lambda_{n-3} & \Lambda_{n-2} \\ . & . & ... & . & . \\ . & . & ... & l & l \\ 0 & 0 & ... & \Lambda_0 & \Lambda_1 \end{bmatrix}_{1,n} .$$

d) Power sum symmetric functions

Analogues of the (commutative) power sum symmetric functions may be defined by:

$$\psi(t)\lambda(-t) = -\frac{d}{dt}\lambda(-t)$$

or

$$\lambda(-t)\xi(t) = -\frac{d}{dt}\lambda(-t).$$

(Note that in the commutative case these definitions coincide.) We may write

$$\psi(t) = \sum_{k \geq 1} t^{k-1} \Psi_k$$

and

$$\xi(t) = \sum_{k \geq 1} t^{k-1} \Xi_k$$

Wenhua Zhao (arXiv:math.CV/0509135v2 9Sep2005) has shown that the Jacobian conjecture is equivalent to a statement about the relation between the Ψ_k and the Ξ_k.